

Torsional Rigidity on Compact Riemannian Manifolds with lower Ricci Curvature Bounds

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Abstract:

In this article we prove a reverse Hölder inequality for the fundamental eigenfunction of the Dirichlet problem on domains of a compact Riemannian manifold with lower Ricci curvature bounds. We also prove an isoperimetric inequality for the torsional rigidity of such domains.

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1 Introduction and Main Results

In 1972, following the spirit of the works of Faber and Krahn [1, 2], Payne and Rayner [3, 4] proved a reverse Hölder inequality for the norms L_1 and L_2 of the first eigenfunction of the Dirichlet problem on bounded domains D of \mathbb{R}^2 :

$$\left(\int_D u_1 dx dy \right)^2 \geq \frac{4\pi}{\lambda_1} \int_D u_1^2 dx dy$$

where λ_1 is the lowest eigenvalue of the fixed membrane problem, and u_1 the corresponding eigenfunction. Equality holds if and only if D is a disc.

The work of Payne and Rayner was generalized by Köhler and Jobin [5] for bounded domains of \mathbb{R}^n , $n \geq 3$. In 1982, G.Chiti [6], generalized the reverse Hölder inequality for the norms L_q and L_p , $q \geq p > 0$ for bounded domains of \mathbb{R}^n , $n \geq 2$.

$$\left(\int_D u^q dx dy \right)^{\frac{1}{q}} \leq K(p, q, \lambda, n) \left(\int_D u^p dx dy \right)^{\frac{1}{p}}$$

Where

$$K(p, q, \lambda, n) = (nC_n)^{\frac{1}{q} - \frac{1}{p}} \lambda^{\frac{(q-p)n}{2pq}} j_{\frac{n}{2}-1,1}^{n(\frac{1}{q} - \frac{1}{p})} \frac{\left(\int_0^1 r^{n-1+q(1-\frac{n}{2})} J_{\frac{n}{2}-1}^q(j_{\frac{n}{2}-1,1} r) dr \right)^{\frac{1}{q}}}{\left(\int_0^1 r^{n-1+p(1-\frac{n}{2})} J_{\frac{n}{2}-1}^p(j_{\frac{n}{2}-1,1} r) dr \right)^{\frac{1}{p}}}$$

This inequality is isoperimetric in the sens that equality holds if and only if D is a ball.

The main ideas of this paper were early investigated in the PHD thesis of H.Hasnaoui [7], the first one was to establish a generalization of Chiti's reverse Hölder inequality for the norms L_q and L_p , $q \geq p > 0$ for Compact Riemannian manifolds with Ricci curvature bounded from below and the second one is a version of the Saint Venant Theorem for such manifolds.

In fact Modern geometric analysts, including Chavel and Gromov, have identified such manifolds as important, and have related the Ricci bound to many estimates of eigenvalues, as well as to other quantities of interest in differential geometry. Thus it is very natural to consider similar questions about the Laplacian and the torsional rigidity in this context. Both are isoperimetric results, is that in both cases the quantity of interest is dominated by the analogous expression on spheres. Much of the background and many results for the spectrum of such manifolds could be found in [8], [9], [10] and more recently in [11]. In 1856, Saint-Venant [12] observed that columns with circular cross-sections offer the greatest resistance to torsion for a given cross-sectional area. This fact was proved a century or so later by Pólya using Steiner symmetrization [13]. We also note the independent proof by Makai in 1963 one can see [14]. Torsional rigidity is a physical quantity of much interest, see for example [15, 16], the recent [17, 18], the more recent [19, 20] and the classical papers of Payne [21, 22], Payne-Weinberger [23]. As we show here, it is also a quantity of much interest for geometers as well. Parts of our work follow an analysis of Ashbaugh and Benguria [24] for subdomains of hemispheres, one can see also [25]. In the sequel we will introduce the main results of this paper:

Let (M, g) be a compact connected Riemannian manifold of dimension $n \geq 1$ without boundary. We denote by

$$R(M, g) = \inf \{ Ric(v, v), \quad v \in UT(M) \},$$

the infimum of the Ricci curvature Ric of (M, g) , here $UT(M)$ is the unit tangent fiber bundle of the manifold (M, g) . Let (\mathbb{S}^n, g^*) be unit the sphere of the space \mathbb{R}^{n+1} , endowed with the induced metric, then $R(\mathbb{S}^n, g^*) = n - 1$. We suppose, as done in [9], that $R(M, g)$ is strictly positive, and normalize the metric g so that $R(M, g) \geq R(\mathbb{S}^n, g^*) = n - 1$. In the sequel, we will denote by $V(M) = \int dv_g$ the volume of (M, g) , where the element of volume is denoted dv_g , ω_n the volume of the unit sphere (\mathbb{S}^n, g^*) and $\beta = V(M)/\omega_n$.

Let D be a connected bounded domain of M with smooth boundary, and D^* the geodesic ball of \mathbb{S}^n centered at the north pole such that

$$Vol(D) = \beta Vol(D^*).$$

We are interested in the comparison of the fundamental solutions u and v of problems (P_1) and (P_2) respectively:

$$(P_1) \begin{cases} \Delta u + \lambda u &= 0 & \text{in } D \\ u &= 0 & \text{on } \partial D, \end{cases}$$

$$(P_2) \begin{cases} \Delta v + \lambda v &= 0 & \text{in } B_{\theta_1(\lambda)} \\ v &= 0 & \text{on } \partial B_{\theta_1(\lambda)}, \end{cases}$$

$B_{\theta_1(\lambda)}$ is the geodesic ball of \mathbb{S}^n centered at the north pole of radius $\theta_1 = \theta_1(\lambda)$, where λ is the first eigenvalue for the Dirichlet problem (P_2) , and Δ denotes indifferently the laplacian operator on M or \mathbb{S}^n .

Let u^* be the decreasing rearrangement of u and u^* the corresponding radial function defined on D^* the geodesic ball of (\mathbb{S}^n, g^*) which has the same relative volume as D , (see section 2 for notations and details).

We prove the following result

Theorem 1.1: Let u be the fundamental eigenfunction of problem (P_1) . Let $p > 0$ and v be the fundamental eigenfunction of problem (P_2) chosen such that

$$(1) \quad \int_D u^p dv_g = \beta \int_{B_{\theta_1(\lambda)}} v^p dv_{g^*}.$$

then, there exists $\theta_2 \in (0, \theta_1)$, such that

$$(2) \quad u^*(\theta) \leq v(\theta), \quad \forall \theta \in [0, \theta_2];$$

$$(3) \quad u^*(\theta) \geq v(\theta), \quad \forall \theta \in [\theta_2, \theta_1].$$

As a consequence of Theorem 1.1, we obtain:

Corollary 1.2: : Chiti's Reverse Hölder Inequality for Compact Manifolds

Let p, q be real numbers such that $q \geq p > 0$, then u and v are related by this inequality

$$(4) \quad \frac{(\int_D u^q dv_g)^{\frac{1}{q}}}{(\int_D u^p dv_g)^{\frac{1}{p}}} \leq \beta^{\frac{1}{q} - \frac{1}{p}} \frac{(\int_{B_{\theta_1(\lambda)}} v^q dv_{g^*})^{\frac{1}{q}}}{(\int_{B_{\theta_1(\lambda)}} v^p dv_{g^*})^{\frac{1}{p}}},$$

with equality if and only if the triplet (M, D, g) is isometric to the triplet (\mathbb{S}^n, D^*, g^*) .

Next, we focus on the torsional rigidity $T(D)$ of the manifold D . Recall that:

$$(5) \quad T(D) = \int_D w dv_g,$$

where w is the smooth solution of the boundary value problem of Dirichlet-Poisson type (w is called the warping function of D)

$$(6) \quad \begin{aligned} \Delta_g w + 1 &= 0 & \text{in } D, \\ w &= 0 & \text{on } \partial D. \end{aligned}$$

We obtain the following result:

Theorem 1.3: : Saint Venant Theorem for Compact Manifolds

Let (M, g) be a complete Riemannian manifold of dimension n , without boundary satisfying $R(M, g) \geq n - 1$, and D a sub manifold of M of the same dimension with smooth boundary, we have

$$(7) \quad T(D) \leq \beta T(D^*).$$

the equality holds if and only if the triplet (M, D, g) is isometric to the triplet (\mathbb{S}^n, D^*, g^*) .

And finally we give a comparison formula for the warping function w which allows us to obtain directly the result of theorem 1.3.

2 Preliminary Tools

Denote by $\lambda(D)$ the first eigenvalue of the Laplacian Δ for the Dirichlet problem on D , and let u be the positive associated eigenfunction. Therefore u satisfies

$$(P_1) \begin{cases} \Delta u + \lambda u &= 0 & \text{in } D \\ u &= 0 & \text{on } \partial D, \end{cases}$$

The variational formula of $\lambda(D)$ is given by

$$\lambda(D) = \inf \left\{ \frac{\int_D |df|^2 dv_g}{\int_D |f|^2 dv_g}; \quad f \not\equiv 0; \quad C^1 - \text{piecewise}; \quad f|_{\partial D} \equiv 0 \right\},$$

where $|df|$ is the Riemannian norm of the differential of f . We have equality if and only if f is of class C^2 and is an eigenfunction associated with the first eigenvalue $\lambda(D)$.

For $0 \leq t \leq \bar{u} = \sup u$, let $D_t = \{x \in D \mid u(x) > t\}$. Define the function

$$(8) \quad V(t) = \int_{D_t} dv_g$$

The co-area formula gives

$$(9) \quad V(t) = \int_{D_t} dv_g = \int_t^{\bar{u}} \int_{\partial D_\tau} \frac{1}{|\nabla u|} d\sigma d\tau.$$

Here $d\sigma$ is the $(n-1)$ -dimensional Riemannian measure in (M, g) . For what follows, we will also denote the $(n-1)$ -dimensional Riemannian measure in (\mathbb{S}^n, g^*) by $d\sigma$. Since D has bounded measure, the above shows that the function

$$(10) \quad t \mapsto \int_{\partial D_t} \frac{1}{|\nabla u|} d\sigma$$

is integrable, and therefore the function V is absolutely continuous. Hence V is differentiable almost everywhere and

$$(11) \quad \frac{dV}{dt} = - \int_{\partial D_t} \frac{1}{|\nabla u|} d\sigma < 0$$

for almost all $t \in [0, \bar{u}]$. The function V is then a non increasing function and has an inverse which we denote by u^* . The function u^* is absolutely continuous. Now, applying the Cauchy-Schwartz inequality, we obtain

$$(12) \quad \left(\int_{\partial D_t} d\sigma \right)^2 \leq \left(\int_{\partial D_t} \frac{1}{|\nabla u|} d\sigma \right) \left(\int_{\partial D_t} |\nabla u| d\sigma \right).$$

Hence

$$(13) \quad -\frac{du^*(s)}{ds} = -\frac{1}{V'(u^*(s))} \leq \frac{\int_{\partial D_{u^*(s)}} |\nabla u| d\sigma}{\left(\int_{\partial D_{u^*(s)}} d\sigma \right)^2}.$$

We now use the following isoperimetric inequality due to M. Gromov [26] which relates the volume of the boundaries of D and D^* .

Lemma 2.1: Under the same hypothesis given above

$$(14) \quad Vol_{n-1}(\partial D) \geq \beta Vol_{n-1}(\partial D^*),$$

where Vol_{n-1} is the $(n-1)$ -dimensional volume relative to g and g^* . Equality holds if and only if the triplet (M, D, g) is isometric to the triplet (\mathbb{S}^n, D^*, g^*) .

Let

$$(15) \quad A(\theta) \equiv s = \beta \omega_{n-1} \int_0^\theta (\sin \tau)^{n-1} d\tau$$

The quantity $\omega_{n-1} \int_0^\theta (\sin \tau)^{n-1} d\tau$ is the n -volume of the geodesic ball of radius θ in \mathbb{S}^n . If we let the function $L(\theta)$ denotes the $(n-1)$ -dimensional volume of the geodesic ball of radius θ , i.e.,

$$(16) \quad L(\theta) = \omega_{n-1} (\sin \theta)^{n-1} = \frac{A'(\theta)}{\beta}.$$

Inequality (14) can then be written as

$$(17) \quad Vol_{n-1}(\partial D) \geq \beta L(\theta(Vol(D))) = \beta \omega_{n-1} (\sin \theta(Vol(D)))^{n-1}$$

where $\theta(s)$ is the inverse function of A defined in (15).

Now, applying inequality (17) to the domain $D_{u^*(s)}$ and combining it with inequality (13), we obtain

$$(18) \quad -\frac{du^*(s)}{ds} \leq (\beta \omega_{n-1})^{-2} (\sin \theta(s))^{2-2n} \int_{\partial D_{u^*(s)}} |\nabla u| d\sigma.$$

We then apply Gauss Theorem to the Dirichlet problem on D_t , using the smoothness of its boundary ∂D_t , we get

$$(19) \quad \int_{\partial D_t} |\nabla u| d\sigma = \int_{D_t} \Delta u dv_g = \lambda \int_{D_t} u dv_g,$$

here we used the fact that the outward normal to D_t is given by $-\frac{\nabla u}{|\nabla u|}$.

Remark $\forall p \geq 0$, we have

$$(20) \quad \int_{D_{u^*(s)}} u^p dv_g = \int_{u^*(s)}^{\bar{u}} \tau^p \int_{\partial D_\tau} \frac{1}{|\nabla u|} d\sigma d\tau = - \int_{u^*(s)}^{\bar{u}} \tau^p V'(\tau) d\tau.$$

The change of variables $\eta = V(\tau)$ gives

$$(21) \quad \int_{D_{u^*(s)}} u^p dv_g = \int_0^s (u^*(\eta))^p d\eta.$$

Finally combining (20) for $p = 1$ with equalities (18) and (19), we obtain the following.

Lemma 2.2: Let u be a solution of problem (P_1) . Then u^* , its decreasing rearrangement, satisfies the integro-differential inequality

$$(22) \quad -u^{*'}(s) \leq \lambda(\beta \omega_{n-1})^{-2} (\sin \theta(s))^{2-2n} \int_0^s u^*(\xi) d\xi.$$

for almost every $s > 0$.

Let $B_{\theta_1(\lambda)}$ be the geodesic ball of \mathbb{S}^n centered at the north pole with radius $\theta_1 = \theta_1(\lambda)$, such that the following problem

$$(P_2) \begin{cases} \Delta v + \lambda v &= 0 & \text{in } B_{\theta_1(\lambda)} \\ v &= 0 & \text{on } \partial B_{\theta_1(\lambda)} \end{cases}$$

has a solution. Let $v > 0$ be the first Dirichlet eigenfunction on $B_{\theta_1(\lambda)}$ then by lemmas 3.1 and 3.2 in [27], we conclude that v depends only of θ and is strictly decreasing on $[0, \theta_1)$. Therefore we denote by $v(\theta)$ the function v . So in polar coordinates the problem (P_2) can be rewritten as

$$(23) \quad -((\sin \theta)^{n-1} v'(\theta))' = \lambda (\sin \theta)^{n-1} v(\theta)$$

in $[0, \theta_1]$. The boundary conditions are $v(0)$ finite and $v(\theta_1) = 0$. Let v^* be the function defined in $[0, A(\theta_1)]$ from the relation

$$(24) \quad v^*(A(\theta)) = v(\theta); \quad \forall \theta \in [0, \theta_1]$$

Integrating the equality (23), we obtain

$$(25) \quad -(\sin \theta)^{n-1} v'(\theta) = \lambda \int_0^\theta (\sin \alpha)^{n-1} v(\alpha) d\alpha$$

Then using (24), we can rewrite the left-hand side of (25) as

$$(26) \quad -(\sin \theta(s))^{n-1} v'(\theta(s)) = -\beta \omega_{n-1} (\sin \theta(s))^{2n-2} v^{*'}(s)$$

for all s in $[0, A(\theta_1)]$. The change of variables $\zeta = A(\alpha)$ in the right-hand side of (25) gives

$$(27) \quad \lambda \int_0^{\theta(s)} (\sin \alpha)^{n-1} v(\alpha) d\alpha = \lambda (\beta \omega_{n-1})^{-1} \int_0^s v^*(\zeta) d\zeta$$

Finally from (26) and (27), we obtain

$$(28) \quad -v^{*'}(s) = \lambda (\beta \omega_{n-1})^{-2} (\sin \theta(s))^{2-2n} \int_0^s v^*(\xi) d\xi$$

3 Chiti's Reverse Hölder Inequality and the Saint-Venant Theorem for Compact Manifolds

3.1 Chiti's Reverse Hölder Inequality

In this section we will prove the extension of the Chiti's Comparison Lemma, given for domains in the case of \mathbb{R}^2 and \mathbb{R}^n in the original papers of Payne-Rayner [3, 4], then extended by Kohler-Jobin [5] and Chiti [6, 28]. In [24] a very general version of the Chiti's Comparison Lemma is available for domains of the sphere \mathbb{S}^{n-1} . Our work deals with the case of a complete Riemannian manifold with bounded Ricci curvature.

Let u^* be the radial function defined in D^* by

$$(29) \quad u^*(\theta) = u^*(A(\theta))$$

CLAIM: we have

$$(30) \quad \text{vol}(D) \geq \beta \text{Vol}(B_{\theta_1(\lambda)})$$

Proof of the Claim: Assume in the contrary that $\beta \text{Vol}(B_{\theta_1(\lambda)}) > \text{vol}(D)$, since $\text{vol}(D) = \beta \text{Vol}(D^*)$, we obtain $\text{Vol}(B_{\theta_1(\lambda)}) > \text{vol}(D^*)$. Then using the fact that the two geodesic balls are centered at the north pole of \mathbb{S}^n , we obtain $D^* \subsetneq B_{\theta_1(\lambda)}$. Finally from domains monotonicity of eigenvalues, we deduce that $\lambda_1(D^*) > \lambda_1(B_{\theta_1(\lambda)}) = \lambda$ which contradicts the result of Theorem 5 in [9].

Now, we will prove the following result which is crucial for the proof of theorem1.1

Lemma 3.1: Let v be the solution of (P_2) chosen such that $v(0) = u^*(0)$. Then

$$(31) \quad v(\theta) \leq u^*(\theta), \quad \forall \theta \in [0, \theta_1].$$

Proof Using (30), two cases occur:

First Case: $\text{vol}(D) = \beta \text{Vol}(B_{\theta_1(\lambda)})$. Then we have $\text{vol}(B_{\theta_1(\lambda)}) = \text{vol}(D^*)$, hence

$$(32) \quad \int_{B_{\theta_1(\lambda)}} |\nabla u^*|^2 dv_{g^*} = \omega_{n-1} \int_0^{\theta_1} \left(\frac{du^*(\theta)}{d\theta} \right)^2 \sin^{n-1} \theta d\theta$$

Using the change of variables $A(\theta) = s$, we obtain

$$(33) \quad \int_0^{\theta_1} \left(\frac{du^*(\theta)}{d\theta} \right)^2 \sin^{n-1} \theta d\theta = \beta \omega_{n-1} \int_0^{A(\theta_1)} (u^{*'}(s))^2 (\sin \theta(s))^{2n-2} ds$$

Then we combine equalities (32) and (33) with lemma2.2 to get

$$(34) \quad \int_{B_\lambda} |\nabla u^*|^2 dv_{g^*} \leq \lambda \beta^{-1} \int_0^{A(\theta_1)} ((-u^{*'}(s) \int_0^s u^*(\xi) d\xi) ds$$

Now, an integration by parts in the second member of the last inequality and the fact that

$$(35) \quad \int_{B_{\theta_1(\lambda)}} u^{*2} dv_{g^*} = \beta^{-1} \int_0^{A(\theta_1)} (u^*(s))^2 ds$$

gives

$$(36) \quad \frac{\int_{B_{\theta_1(\lambda)}} |\nabla u^*|^2 dv_{g^*}}{\int_{B_{\theta_1(\lambda)}} u^{*2} dv_{g^*}} \leq \lambda$$

Using that λ is the minimum of the Rayleigh quotient on $B_{\theta_1(\lambda)}$, it follows that this minimum is achieved for u^* , and so u^* is indeed an eigenfunction associated with λ on $B_{\theta_1(\lambda)}$. Now, using the simplicity of the fundamental eigenvalue and the hypothesis of our Lemma, we get $u^* = v$.

Second Case: $\text{vol}(D) > \beta \text{Vol}(B_{\theta_1(\lambda)})$.

In one hand, we have $u^*(A(\theta_1)) > 0$ while $v^*(A(\theta_1)) = 0$. In the other hand

$$(37) \quad u^*(0) = u^*(0) = v(0) = v^*(0)$$

then, we can find a constant $c \geq 1$, such that

$$(38) \quad c u^*(s) \geq v^*(s) \quad \forall s \in [0, A(\theta_1)]$$

Let c' be the constant defined by

$$(39) \quad c' = \inf\{c \geq 1 ; \quad c u^*(s) \geq v^*(s), \quad \forall s \in [0, A(\theta_1)]\}$$

Then by the definition of c' , we can find $s_1 \in [0, A(\theta_1)]$, such that $c' u^*(s_1) = v^*(s_1)$.

We define now the following function

$$h(s) = \begin{cases} c' u^*(s); & \text{if } s \in [0, s_1] \\ v^*(s); & \text{if } s \in [s_1, A(\theta_1)] \end{cases}$$

The properties of u^* and v^* imply that h is monotonically decreasing and $h(A(\theta_1)) = 0$. Further, in virtue of (22) and (28), we easily see that

$$(40) \quad -h'(s) \leq \lambda(\beta \omega_{n-1})^{-2} (\sin \theta(s))^{2-2n} \int_0^s h(\xi) d\xi$$

for almost all $s \in [0, A(\theta_1)]$.

Now, let \mathfrak{h} be a radial function defined in $B_{\theta_1(\lambda)}$ by

$$(41) \quad \mathfrak{h}(\theta) = h(A(\theta))$$

then, \mathfrak{h} is an admissible function for the Rayleigh quotient on $B_{\theta_1(\lambda)}$. From this we proceed exactly as in the proof of inequality (39), we obtain

$$(42) \quad \frac{\int_{B_{\theta_1(\lambda)}} |\nabla \mathfrak{h}|^2 dv_{g^*}}{\int_{B_{\theta_1(\lambda)}} \mathfrak{h}^2 dv_{g^*}} \leq \lambda$$

and by the definition of $B_{\theta_1(\lambda)}$ it follows that \mathfrak{h} is an eigenfunction associated with λ on $B_{\theta_1(\lambda)}$. Then \mathfrak{h} is a multiple of v and so, from the definition of h , it follows that $h = v^*$ and $c' u^*(s) = v^*(s)$ for $0 \leq s \leq s_1$. Since $u^*(0) = v^*(0)$, then $c' = 1$ and $u^*(s) \geq v^*(s)$ for all $0 \leq s \leq A(\theta_1)$. The proof of the lemma is thereby complete.

Proof of theorem 1.1 The normalization condition (1) is equivalent to

$$(43) \quad \int_0^{vol(D)} (u^*(s))^p ds = \int_0^{A(\theta_1)} (v^*(s))^p ds$$

Since the functions u^* and v^* are nonnegative, and $A(\theta_1) \leq vol(D)$ (see (30)), it is then clear that

$$(44) \quad \int_0^{A(\theta_1)} (u^*(s))^p ds \leq \int_0^{A(\theta_1)} (v^*(s))^p ds$$

We will first prove that $v^*(0) \geq u^*(0)$. Assume that $v^*(0) < u^*(0)$. In this case, $\exists \kappa > 1$, such that $\kappa v^*(0) = u^*(0)$. By Lemma 3.1, we have

$$(45) \quad \kappa v^*(s) \leq u^*(s) \quad \forall s \in [0, A(\theta_1)]$$

Therefore

$$\kappa^p \int_0^{A(\theta_1)} (v^*(s))^p ds \leq \int_0^{A(\theta_1)} (u^*(s))^p ds$$

Combining this inequality with (44) leads to $\kappa^p \leq 1$, which is a contradiction.

Suppose now that $v^*(0) = u^*(0)$. From (43) and Lemma 3.1, we obtain

$$(46) \quad \int_0^{vol(D)} (u^*(s))^p ds = \int_0^{A(\theta_1)} (v^*(s))^p ds \leq \int_0^{A(\theta_1)} (u^*(s))^p ds$$

This means $\int_{A(\theta_1)}^{vol(D)} (u^*(s))^p ds \leq 0$, and since $u^* > 0$ in $(0, vol(D))$, we have $vol(D) = A(\theta_1)$.

Then $v = u^*$, and the statements of the theorem are evident.

Now, we treat the case $v^*(0) > u^*(0)$.

In this case $vol(D) > A(\theta_1)$. Therefore, $v^*(A(\theta_1)) = 0$ and $u^*(A(\theta_1)) > 0$. Now, by the continuity of v^* and u^* , we see that $v^*(s) > u^*(s)$ in a neighborhood of 0, and there exists $s_1 \in (0, A(\theta_1))$ such that $v^*(s_1) = u^*(s_1)$. Choose s_1 to be the largest such number with the additional property that $u^*(s) \leq v^*(s)$ for all $s \in [0, s_1]$. By the definition of s_1 , there is an interval immediately to the right of s_1 on which $u^*(s) > v^*(s)$. We will now show that $u^*(s) > v^*(s)$ for all $s \in (s_1, A(\theta_1)]$. If not, there exists $s_2 \in (s_1, A(\theta_1))$ such that $u^*(s_2) = v^*(s_2)$ and $u^*(s) > v^*(s)$ for all $s \in (s_1, s_2)$. In this case, we can define the function

$$\varphi(s) = \begin{cases} v^*(s), & \text{for } s \in [0, s_1] \cup [s_2, A(\theta_1)], \\ u^*(s), & \text{for } s \in [s_1, s_2] \end{cases}$$

It follows from (22) and (28) that φ satisfies

$$(47) \quad -\varphi'(s) \leq \lambda (\beta \omega_{n-1})^{-2} (\sin \theta(s))^{2-2n} \int_0^s \varphi(\xi) d\xi$$

From φ define a radial function in $B_{\theta_1(\lambda)}$ by

$$(48) \quad \Phi(\theta) = \varphi(A(\theta))$$

Then Φ is an admissible function for the Rayleigh quotient on $B_{\theta_1(\lambda)}$. Using this fact we proceed exactly as in the proof of inequality (35), we obtain

$$(49) \quad \frac{\int_{B_{\theta_1(\lambda)}} |\nabla \Phi|^2 dv_{g^*}}{\int_{B_{\theta_1(\lambda)}} \Phi^2 dv_{g^*}} \leq \lambda$$

It follows that the Rayleigh quotient of Φ is equal to λ therefore Φ is an eigenfunction for λ , Consequently, $u^* = v^*$ and so $u^*(s) = v^*(s)$ in $[s_1, s_2]$ which contradicts the maximality of s_1 and hence completes the proof of the theorem.

Proof of Corollary1.2: Chiti's Reverse Hölder Inequality

For $p > 0$, we choose v so that the condition (1) is satisfied. Now, if we extend the function v^* by zero in $[A(\theta_1), vol(D)]$, we obtain

$$(50) \quad \int_0^s (u^*(\eta))^p d\eta \leq \int_0^s (v^*(\eta))^p d\eta \quad \forall s \in [0, vol(D)]$$

To see (50), we note that Theorem1.1 implies the following:

If $s \in [0, s_1]$, then

$$u^*(\eta) \leq v^*(\eta), \quad \forall \eta \in [0, s]$$

hence

$$\int_0^s (u^*(\eta))^p d\eta \leq \int_0^s (v^*(\eta))^p d\eta$$

If $s \in [s_1, vol(D)]$, then

$$\begin{aligned} \int_0^s (u^*(\eta))^p d\eta &= \int_0^{vol(D)} (u^*(\eta))^p d\eta - \int_s^{vol(D)} (u^*(\eta))^p d\eta \\ &\leq \int_0^{vol(D)} (v^*(\eta))^p d\eta - \int_s^{vol(D)} (v^*(\eta))^p d\eta \\ &= \int_0^s (v^*(\eta))^p d\eta \end{aligned}$$

We complete the argument using the following result

Lemma 3.2: [29] Let R, p, q be real numbers such that $0 < p \leq q$, $R > 0$; and f, g real functions in $L^q([0, R])$. If the decreasing rearrangements of f and g satisfy the following inequality:

$$\int_0^s (f^*)^p dt \leq \int_0^s (g^*)^p dt, \quad \forall s \in [0, R],$$

then

$$\int_0^R f^q dt \leq \int_0^R g^q dt.$$

From this, it is clear, that for all $q \geq p$,

$$(51) \quad \int_0^{\text{vol}(D)} (u^*(\eta))^q d\eta \leq \int_0^{\text{vol}(D)} (v^*(\eta))^q d\eta = \int_0^{A(\theta_1)} (v^*(\eta))^q d\eta$$

Finally, combining this inequality and equality (1), we obtain the desired result.

Now assume that we have equality in (4), from the normalization for the function v given in (1), we deduce that for all $p > 0$, $\int_D u^p dv_g = \beta \int_{B_{\theta_1(\lambda)}} v^p dv_{g^*}$. Hence $\text{vol}(D) = \beta \text{vol}(B_{\theta_1(\lambda)})$, and since D^* and $B_{\theta_1(\lambda)}$ are geodesic balls of \mathbb{S}^n centered both at the north pole with the same volume, it yields that $D^* = B_{\theta_1(\lambda)}$. By hypothesis λ is the fundamental eigenvalue of $B_{\theta_1(\lambda)}$, hence of D^* , thus we obtain that $\lambda_1(D) = \lambda_1(D^*) = \lambda$ and this is possible if and only if the triplet (M, D, g) is isometric to the triplet (\mathbb{S}^n, D^*, g^*) , (one can see Theorem 5 in [9]). The proof of Corollary 1.2 is thereby complete.

3.2 Saint-Venant Theorem

Let (M, g) be a compact Riemannian manifold of dimension n , without boundary satisfying $R(M, g) \geq n - 1$, and let D a bounded connected domain of M with smooth boundary. We are interested in the following geometric quantity

$$(52) \quad T(D) = \int_D w dv_g$$

where w is the smooth solution of the boundary value problem of Dirichlet-Poisson type

$$(53) \quad \begin{aligned} \Delta_g w + 1 &= 0 \text{ in } D, \\ w &= 0 \text{ on } \partial D \end{aligned}$$

The geometric quantity $T(D)$ is called the “torsional rigidity of D ”, and it is customary to call the solution w of (53) the warping function. In theorem 1.3 we will give explicit upper bounds for the torsional rigidity $T(D)$ which amounts to a version of the Saint-Venant Theorem for compact manifolds. Let $C_0^\infty(D)$ denotes the space of C^∞ functions with compact support in D . We define the Sobolev space $H_0^1(D)$ as the closure of $C_0^\infty(D)$ in $H^1(M)$ the space of square integrable functions with a square integrable weak derivatives. The variational formulation for $T(D)$ is given by

$$(54) \quad \frac{1}{T(D)} = \inf \left\{ \Phi(f) = \frac{\int_D |\nabla u|^2 dv_g}{(\int_D u dv_g)^2}, \quad f \in H_0^1(D), \quad f \not\equiv 0 \right\}$$

Indeed, by the scaling property of the functional, $\Phi(cf) = \Phi(f)$ for all $c > 0$, one can reformulate the minimizing problem of the functional Φ as a minimizing problem of the functional $\int_D |\nabla f|^2 dv_g$ subject to the constraint $\int_D f dv_g = 1$. By the above mentioned Lagrange multipliers theorem, this gives the existence of the Lagrange multiplier ρ , such that for any $h \in H_0^1(D)$

$$(55) \quad \int_D \langle \nabla f, \nabla h \rangle dv_g = \rho \int_D h dv_g$$

Hence, f is a weak solution of the equation

$$(56) \quad \begin{aligned} \Delta f &= -\rho & \text{in } D, \\ f &= 0 & \text{on } \partial D \end{aligned}$$

By standard regularity results, f is unique and smooth. Then $w = \rho^{-1}f$ is also a critical point of Φ . Finally the fact that $\Phi(w) = \frac{1}{T(D)}$ proves the equality.

We will now give the proof of the Saint-Venant Theorem.

Proof of Theorem 1.3

The proof follows the steps of Talenti's method (one can see [30]) tailored to our setting. For $0 < t \leq m = \sup\{w(x), x \in D\}$, we define the set

$$(57) \quad D_t = \{x \in D; w(x) > t\}$$

and introduce the following functions

$$(58) \quad V(t) = \int_{D(t)} dv_g, \quad \Psi(t) = \int_{D(t)} |\nabla w|^2 dv_g$$

The smooth co-area formula gives

$$(59) \quad V(t) = \int_t^m \int_{\partial D_\tau} \frac{1}{|\nabla w|} d\sigma d\tau, \quad \Psi(t) = \int_t^m \int_{\partial D_\tau} |\nabla w| d\sigma d\tau$$

Then differentiating (58) with respect to t , one obtain

$$(60) \quad V'(t) = - \int_{\partial D_t} \frac{1}{|\nabla w|} d\sigma, \quad \Psi'(t) = - \int_{\partial D_t} |\nabla w| d\sigma$$

Let w^* be the inverse function of V , w^* the radial function defined in D^* by $w^*(\theta) = w^*(A(\theta))$, and

$$(61) \quad V_\star(t) = \int_{D_t^\star} dv_{g^\star}, \quad \Psi_\star(t) = \int_{D_t^\star} |\nabla w^\star|^2 dv_{g^\star}$$

Since w^\star is a radial decreasing function, its level sets D_t^\star are geodesic balls with radius $r(t) = w^{\star-1}(t)$. Therefore

$$(62) \quad V_\star(t) = \omega_{n-1} \int_0^{r(t)} \sin^{n-1} \theta d\theta$$

and

$$(63) \quad \Psi_*(t) = \omega_{n-1} \int_0^{r(t)} \left(\frac{dw^*}{d\theta} \right)^2 \sin^{n-1} \theta d\theta$$

Differentiating with respect to t , we get

$$(64) \quad V'_*(t) = -|\nabla w^*|^{-1}(t) \omega_{n-1} \sin^{n-1} r(t) = -|\nabla w^*|^{-1}(t) \int_{\partial D_t^*} d\sigma$$

and

$$(65) \quad \Psi'_*(t) = -|\nabla w^*|(t) \omega_{n-1} \sin^{n-1} r(t) = -|\nabla w^*|(t) \int_{\partial D_t^*} d\sigma$$

On one hand multiplying (64) by (65), we obtain

$$(66) \quad \left(\int_{\partial D_t^*} d\sigma \right)^2 = V'_*(t) \Psi'_*(t)$$

In the other hand the Cauchy-Schwartz inequality gives

$$(67) \quad \left(\int_{\partial D_t} d\sigma \right)^2 \leq V'(t) \Psi'(t)$$

From the fact that $V_*(t) = \beta^{-1} A(r(t)) = \beta^{-1} V(t)$, we have $V'(t) = \beta V'_*(t)$, then we use Gromov's isoperimetric inequality (14), (66) and (67) to obtain

$$(68) \quad \beta \Psi'_*(t) \geq \Psi'(t)$$

Now by integrating this inequality from 0 to m and using the fact that $\beta \Psi_*(m) = 0 = \Psi(m)$, we get

$$(69) \quad \beta \int_{D^*} |\nabla w^*|^2 dv_{g^*} = \beta \Psi_*(0) \leq \Psi(0) = \int_D |\nabla w|^2 dv_g$$

Next, the fact that

$$(70) \quad \beta \int_{D^*} w^* dv_{g^*} = \int_D w dv_g$$

and the variational formulation given in (54), it yields

$$(71) \quad \frac{1}{T(D)} = \frac{\int_D |\nabla w|^2 dv_g}{\left(\int_D w dv_g \right)^2} \geq \frac{1}{\beta} \frac{\int_{D^*} |\nabla w^*|^2 dv_{g^*}}{\left(\int_{D^*} w^* dv_{g^*} \right)^2} \geq \frac{1}{\beta T(D^*)}$$

In the case of equality, going back to (68), which we integrate, then, we use the facts that $\beta \Psi_*(m) = \Psi(m)$ and $\beta \Psi_*(0) = \Psi(0)$, we prove that $\beta \Psi'_*(t) = \Psi'(t)$ for all $0 < t \leq m$. Next, we use the last equality in (66) and (67), we get

$$(72) \quad \text{vol}_{n-1}(\partial D_t^*) = \int_{\partial D_t^*} d\sigma = \int_{\partial D_t} d\sigma = \text{vol}_{n-1}(\partial D_t)$$

to achieve the proof, we apply once again Gromov's isoperimetric inequality.

In the sequel, we will give a comparison theorem for the warping function in the case of a smooth compact Riemannian manifold. This theorem is based on a method of Talenti ([30]).

Theorem 3.3: Let (M, g) be a compact Riemannian manifold of dimension n , without boundary satisfying $R(M, g) \geq n - 1$, and D a bounded connected domain of M with smooth boundary. Let v be the radial function defined in D^* by

$$(73) \quad v(\theta) = \int_{\theta}^{\theta_0} \left(\int_0^{\delta} \sin^{n-1} \tau d\tau \right) \sin^{1-n} \delta d\delta$$

which is the solution to the problem

$$(74) \quad \begin{aligned} \Delta_g v + 1 &= 0 \text{ in } D^*, \\ v &= 0 \text{ on } \partial D^* \end{aligned}$$

Then w^* , the symmetrized function of w satisfies

$$(75) \quad w^* \leq v \quad \text{a.e. in } D^*$$

We obtain the equality if and only if the triplet (M, D, g) is isometric to the triplet (\mathbb{S}^n, D^*, g^*)

Proof

Let f be a test function in the weak formulation of our problem defined on D by

$$(76) \quad f(x) = \begin{cases} w(x) - t, & \text{if } w(x) > t \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq t < m$. We introduce the function defined by

$$(77) \quad \Psi(t) = \int_{D_t} |\nabla w|^2 dV_g$$

Then

$$(78) \quad \Psi(t) = \int_{w>t} (w - t) dV_g$$

The function Ψ is decreasing in t then, for $h > 0$ we have

$$\frac{\Psi(t) - \Psi(t+h)}{h} = \int_{w>t+h} dV_g + \int_{t<w\leq t+h} \left(\frac{w-t}{h} \right) dV_g$$

Letting h go to zero, we obtain, for the right derivative of $\Psi(t)$,

$$(79) \quad -\Psi'_+(t) = \int_{w>t} dV_g \quad \text{a.e. } t > 0$$

The same computation gives the same equality for the left derivative of $\Psi(t)$. Therefore

$$(80) \quad 0 \leq -\Psi'(t) = V(t)$$

Next, we use the Cauchy-Schwarz inequality

$$\left(\frac{1}{h} \int_{t < w \leq t+h} |\nabla w| dV_g \right)^2 \leq \left(\frac{1}{h} \int_{t < w \leq t+h} |\nabla w|^2 dV_g \right) \left(\frac{1}{h} \int_{t < w \leq t+h} dV_g \right)$$

Thus, letting $h \rightarrow 0$ and using (80), we obtain

$$(81) \quad \left(-\frac{d}{dt} \int_{f>t} |\nabla w| dV_g \right)^2 \leq (-V'(t)V(t))$$

Using the co-area formula, it yields

$$(82) \quad -\frac{d}{dt} \left(\int_{w>t} |\nabla w| dV_g \right) = \int_{\partial D_t} d\sigma, \quad \text{a.e. } t > 0$$

Next by M.Gromov's isoperimetric inequality (14), we have

$$(83) \quad \int_{\partial D_t} d\sigma \geq \beta \int_{\partial D_t^*} d\sigma = \beta \omega_{n-1} \sin^{n-1} r(t) = A'(r(t))$$

Combining this with (82) and (81), we get

$$(84) \quad -V'(t)V(t) \geq (A'(r(t)))^2$$

for almost every t in $(0, m)$. Using the fact that $A(r(t)) = V(t)$ and $r^{-1}(\theta) = w^*(\theta)$, we get

$$(85) \quad -w^{*\prime}(\theta) \leq \frac{A(\theta)}{A'(\theta)}$$

Now, for $\theta \in (0, \theta_0)$, integrating this inequality from θ to θ_0 , we obtain

$$(86) \quad w^*(\theta) \leq \int_{\theta}^{\theta_0} \frac{A(\tau)}{A'(\tau)} d\tau = v(\theta)$$

which is the desired result.

Now assume that we have equality in (7), integrating this equality, we get

$$(87) \quad T(D) = \int_D w dV_g = \beta \int_{D^*} w^* dV_{g^*} = \beta \int_{D^*} v dV_{g^*}$$

Finally by applying the Saint-Venant Theorem, we deduce that the triplet (M, D, g) is isometric to the triplet (\mathbb{S}^n, D^*, g^*) , which completes the proof of the theorem.

Remark 3.4: As a consequence of the result above, we obtain after integrating inequality (75):

$$(88) \quad T(D) = \int_D w dv_g = \beta \int_{D^*} w^* dv_{g^*} \leq \beta \int_{D^*} v dv_{g^*} = T(D^*)$$

Which is the result of Theorem 1.3.

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